Equipartition and Rate of Energy Exchanges in a Model of a Radiant Cavity

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Results of calculations on a model of a radiant cavity, performed in order to explore the relation between stochasticity and geometrical structure of phase space, are presented. The rate of energy exchanges, as indicator of stochasticity, is found to be quite effective. Furthermore, a trend to equipartition for such a quantity is observed at increasing energy, and this implies an increasing rigidity of high harmonic modes also in the stochastic regime of motion. Such a feature may be correlated to the shape of the spectrum which characterizes the radiant cavity with respect to nonlinear chains.

KEY WORDS: Energy exchanges; equipartition; nonlinear dynamics; radiant cavity; stochasticity.

1. INTRODUCTION

There exists a broad spectrum of attempts[1,2] to join dynamics and the geometry of phase space of dynamical systems, in order to have a better insight into the mechanism of the onset of stochasticity. A number of unsolved questions can profit from this: for instance, the relevance of KAM theory in the thermodynamic limit and the relations between thermodynamic and continuum limits. In refs. 2, classical examples of finite one-dimensional nonlinear chains have been studied by introducing, among other quantities, the rate of energy exchanges of normal modes, and it has been shown that a strict correlation exists between the behavior of this parameter and stochasticity; moreover, such a correlation may be interpreted in terms of the geometrical complexity of phase space. Since the

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present work extends the approach of refs. 2 in a qualitatively different context, some notations and results will be recalled here.

For a Hamiltonian system consisting of N harmonic oscillators one introduces normal mode coordinates $\{a_1,..., a_N; \dot{a}_1,..., \dot{a}_N\}$ and action–angle variables $\{J_1,..., J_N; \theta_1,..., \theta_N\}$ in such a way that the harmonic part H^0 of the Hamiltonian function may be written as

$$H^0 = \sum_{n=1}^{N} \frac{1}{2}(\dot{a}_n^2 + \omega_n^2 a_n^2) = \sum_{n=1}^{N} \omega_n J_n$$

The system with Hamiltonian H^0 is integrable, i.e., there exist N integrals of motion, the energies of the modes, which are in convolution, and the whole of phase space is continuously foliated by the invariant surfaces (N-tori) determined by these integrals. Since the energy of the nth mode is $E_n = \omega_n J_n$, it follows that $\dot{E}_n = 0$ and $\dot{J}_n = 0$ for the harmonic chain. For a nonlinear system of coupled oscillators with Hamiltonian $H = H^0 + V$, the variations of the harmonic actions are connected to the deformation or breaking of these surfaces, and it may be conjectured that the increasing complexity of the trajectories in phase space may be used as a stochastic parameter. The onset of stochasticity, in particular, should influence suitable quantities based on the time behavior of the actions, or, more generally, on the geometrical features of the trajectories.

Such conjecture has been checked in refs. 2 with several techniques, and, in particular, by studying the quantity $\mathcal{X}_T = \text{Var}(\langle |\dot{J}_1| \rangle_T,..., \langle |\dot{J}_N| \rangle_T)$, where $\langle \cdot \rangle_T$ denotes the time average up to time T, namely, for any integrable function $f(t)$

$$\langle f \rangle_T = \frac{1}{T} \int_0^T f(t)\, dt \tag{1.1}$$

and $\text{Var}(\cdots)$ denotes the variance of its arguments. The $|\dot{J}_n|$ are the absolute values of the time derivatives of the *harmonic* action variables J_n. For the harmonic chain, since $\dot{J}_n \equiv 0$, one has $\mathcal{X}_T \equiv 0$. For an anharmonic chain, in the ordered region of phase space the value of $\mathcal{X}_T$ depends on the initial conditions. What is important for our purposes is that $\mathcal{X}_T$ proves to be very sensitive to the stochastic transition and may be used in turn as an efficient indicator of stochasticity. This means the following: consider a system such as the Lennard-Jones chain, which is known to undergo a transition to stochasticity at a value u^c of the specific energy u, in the sense that for $u > u^c$ the motion is chaotic. Then, coherently, it happens that when $u > u^c$ one finds $\mathcal{X}_T \to 0$ for $T \to \infty$ independently of the initial conditions. This limit for $\mathcal{X}_T$ and the relation $\langle |\dot{E}_n| \rangle = \omega_n \langle |\dot{J}_n| \rangle$ imply

$$\langle |\dot{J}_n| \rangle = \text{const}, \qquad \langle |\dot{E}_n| \rangle \propto \omega_n \tag{1.2}$$

where $\langle \cdot \rangle = \lim_{T \to \infty} \langle \cdot \rangle_T$. As usual in these studies, the limit to ∞ is to be understood as an inference from clear numerical evidence.

The results summarized by (1.2) admit a geometrical interpretation: in the highly stochastic domain, mean equipartition among the $\langle |\dot{J}_n| \rangle$ means that the orbits run with the same mean velocity in the whole of phase space spanned by the action variables, and this is an index of homogeneous disorder.

In refs. 2, the picture of high complexity of the geometry of phase space has been confirmed through the study of other observables, namely the variance of the curvature along the trajectory and the variance of the microcanonical density. The analysis of these further quantities gave evidence of the following phenomena: first, the thermalization corresponds to a drastical increase of the geometrical complexity of orbits; second, a domain of *low stochasticity* may be detected, where the system behaves as if there were a trapping effect of residual invariant surfaces on trajectories during *finite* time observations. We shall not be concerned here with curvature and microcanonical density, because of the neater numerical evidence obtained through the rate of energy exchanges, which is also easier to compute.

Since for anharmonic chains the frequency spectrum is bounded as $N \to \infty$ by a finite value ω_∞, formula (1.2) does not present any *a priori* difficulty in the thermodinamic limit. Such considerations cannot be trivially extended to systems with unbounded spectrum. Typically, for a field where $\omega_n \propto n$, formula (1.2) would imply a divergence in the energy exchanges among the modes. The freezing of high modes appears in such a case as a "physical" necessity, and one may wonder which form the dependence of the thermalized $\langle |\dot{E}_n| \rangle$ on the frequencies will assume.

Our purpose here consists precisely in extending the approach of refs. 2 to an infinite system with an unbounded spectrum. We have chosen the model of a radiant cavity introduced in ref. 3 and studied, for example, in refs. 4–6. The interesting features of this system are the following: (1) it admits a Hamiltonian description, which enables systematic analogies and comparisons with nonlinear chains; (2) it exhibits the phenomenon of a stochastic transition characterized by the existence of thresholds in the energies of the modes; (3) it offers certain possibilities of studying the limiting properties when the number N of degrees of freedom goes to infinity.

Statements 1–3 above should be taken, at this point, in some naive sense. As usual for numerical studies, the definite meaning and the semantic bounds of such key words as equipartition, stochastic transition, etc., will arise quite naturally from the context. In particular, the main points investigated here, i.e., the influence of the shape of the spectrum on the

relaxation and the relation between the dynamical regime of motion and the geometrical features of phase space, appear from our considerations to be meaningful independently of a complete mathematical formulation of the statistical mechanics of infinite systems.

The problem above, indeed, will be approached in the present paper mainly through numerical methods. The behavior of each mode will be considered with respect to both the energy distribution and the rate of energy exchanges. This last is clearly the main object of investigation, while computations on the energy distribution are essentially meant to recover known criteria of stochasticity and to support our results by methods which are widely checked. The time scale of different observed phenomena will enter as a fundamental observable in itself.

Obviously, the numerical calculations are performed for a system with a finite number of degrees of freedom (mostly $N = 25$). However, at the highest values of energy and evolution time chosen for the experiments, the high modes (e.g., those with $n > 19$) are still frozen. Under such conditions, as it will be explained in the next section, our experiments simulate a system with infinitely many degrees of freedom.

The reason why we study the behaviors of several modes separately, instead of their variance as in refs. 2, is that here, in principle, there are infinitely many modes and the value of the variance is drastically influenced by the maximal N chosen. In any case, the analysis on individual modes enables systematic comparisons with formula (1.2). What actually results from our computations is that, for *thermalized* modes, one has

$$\langle |\dot{E}_n| \rangle = \text{const}, \qquad \langle |\dot{J}_n| \rangle \propto 1/\omega_n \tag{1.3}$$

The mutual relation between the limit for $T \to \infty$ and the thermalization process remains an open problem, as it will be discussed in Section 4.

Now, taking into account the recalled geometrical interpretation of (1.2), the comparison with (1.3) clearly indicates that, when the spectrum is unbounded, in the very stochastic domain there exists a clear qualitative difference in the geometric structure of the space: in this case, in fact, the mean absolute velocity of a harmonic action is inversely proportional to its frequency. So, not only are high modes more and more difficult to excite, but, also when thermalized, their motions preserve a sort of "rigidity" of the space: $\langle |\dot{J}_n| \rangle$ represents indeed the absolute velocity of the nth radius of the unperturbed torus, which is identically 0 for the harmonic system.

In Section 2 we present the model along the lines of ref. 5, which enables us to avoid the reduction to a finite system (even if we treat only a finite number of observables). In Section 3 we describe the numerical experiments, and finally in Section 4 comments and conclusions are given.

2. THE MODEL

The physical system we are interested in was introduced and described in ref. 3. It consists of a uniformly charged plate which moves between two parallel and perfectly reflecting mirrors. The motion is parallel to the mirrors, midway between them. Taking the YZ axis on the plate and the Z axis along the motion, and considering the Coulomb gauge, one can write the equations for the z component of the vector potential $A(x, t)$ and for the coordinate $z(t)$ of a reference point on the plane:

$$\frac{\partial^2 A_z}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 A_z}{\partial t^2} = -\frac{4\pi}{c}\sigma\delta(x)\,\dot{z}$$

$$m\ddot{z} = -\frac{\sigma}{c}\frac{\partial A_z}{\partial t}(0, t) + F(z)$$

$$(2.1)$$

$$A_z(-l, t) = A_z(l, t) = 0 \tag{2.2}$$

Here c is the velocity of light, m and σ are the mass and the charge densities, $2l$ is the distance between the mirrors, and the Dirac δ function is connected to the "infinite" ratio between l and the thickness of the plate. $F(z)$ represents a mechanical restoring force, which throughout this work will be assumed to be

$$F(z) = -m\alpha z^3 \tag{2.3}$$

An additional linear term of the form $-m\omega_0^2 z$ has been shown in ref. 5 to be qualitatively uneffective. Equations (2.1) will be solved with the initial conditions discussed in the next section.

The field A_z will now be transformed into an infinite system of coupled oscillators, by introducing the normal mode coordinates a_n. The functions $\cos\omega_n x/c$, where $\omega_n = n\pi c/2l$ and n is odd, are a complete basis for summable functions satisfying the boundary conditions (2.2). Therefore, one can expand the field as

$$A_z(x, t) = \sum_{n=1}^{\infty}{}' a_n(t)\,u_n(x)$$

where $u_n(x) = 2c(\pi/l)^{1/2}\cos\omega_n x/c$ and $\sum'$ means the sum over the odd terms only. One obtains then the equations

$$\ddot{a}_n + \omega_n^2 a_n = 2(\pi/l)^{1/2}\sigma\dot{z} \qquad (n = 1, 3, 5,...)$$

$$\ddot{z} = -2\left(\frac{\pi}{l}\right)^{1/2}\frac{\sigma}{m}\sum_{n=1}^{\infty}{}' \dot{a}_n - \alpha z^3$$

$$(2.4)$$

The total energy of this system is given by

$$E = \frac{1}{2} m\dot{z}^2 + V(z) + \sum_{n=1}^{\infty}{}' E_n, \qquad E_n = \frac{1}{2}(\dot{a}_n^2 + \omega_n^2 a_n^2) \qquad (2.5)$$

Note that the variables used here are not canonical; field-charge interaction terms appear, however, if such variables are used. With the standard technique already illustrated in ref. 5, one expresses the a_n as a sum of a particular integral and the solutions a_n^0 of the associated homogeneous equations, the initial conditions being imposed on the latter. Then one substitutes the a_n in the second of (2.4), obtaining an integro-differential equation for $z(t)$. The simple identity

$$\sum_{n=1}^{\infty}{}' \int_{-a}^{a} f(t) \cos \frac{n\pi t}{2a} = \frac{a}{2} [f(0^+) + f(0^-)]$$

allows one to rewrite the system (2.4) in the form

$$\ddot{a}_n + \omega_n^2 a_n = 2(\pi/l)^{1/2} \sigma \dot{z} \qquad (n = 1, 3, 5, ...)$$

$$(2.6)$$

$$\ddot{z} = -2\left(\frac{\pi}{l}\right)^{1/2} \frac{\sigma}{m} \sum_{n=1}^{\infty}{}' \dot{a}_n^0(t) - \frac{2\pi\sigma^2}{mc}\left[\dot{z}(t) + 2\sum_{k=1}^{I(t)} (-1)^k \dot{z}\left(t - 2\frac{kl}{c}\right)\right] - \alpha z^3$$

where $I(t)$ is the integer part of $tc/2l$ and $t > 0$ (modifications for $t < 0$ are obvious).

The systems (2.4) and (2.6) are equivalent, but as far as one considers only a finite number N of modes (M of them initially excited), the approximate systems behave differently: system (2.4) is conservative at any finite N, but the behavior of the single nth mode depends on N; on the contrary, the N approximation to the system (2.6) is not a conservative one, because the terms in $\dot{z}$ contain contributions from all the modes, but it has the advantage that the single modes may be computed independently from each other and from N. Since we are concerned with the behavior of single modes, we proceed by solving system (2.6). (In our computations modes with $n > N$ are always frozen, i.e., they do not absorb energy, so that the N-dimensional approximation is also conservative).

With the rescalings

$$\tilde{\omega}_n = \omega_n \frac{l}{c} = n\frac{\pi}{2}, \qquad \tilde{\alpha} = \alpha \frac{l^4}{c^2}$$

$$(2.7)$$

$$\tilde{E} = \frac{E}{mc^2}, \qquad \tilde{E}_n = \frac{E_n}{mc^2}$$

and introducing the dimensionless parameters

$$\gamma = \frac{2\pi l \sigma^2}{mc^2} \tag{2.8a}$$

$$\varepsilon = \frac{1}{\gamma} \frac{l}{c} \left(\frac{E\alpha}{m} \right)^{1/4} = \frac{1}{\gamma} (\tilde{E}\tilde{\alpha})^{1/4} \tag{2.8b}$$

and the new dimensionless variables

$$\tau = \frac{c}{l} t, \qquad b_n = \frac{a_n}{l(m\tilde{E})^{1/2}} \gamma^2, \qquad w = \frac{z}{l(\tilde{E})^{1/2}} \gamma^2 \tag{2.9}$$

one obtains the final equations

$$\ddot{b}_n + \tilde{\omega}_n^2 b_n = (2\gamma)^{1/2} \dot{w}$$

$$\ddot{w} = -(2\gamma)^{1/2} \sum_{n=1}^{\infty}{}' \dot{b}_n^0(\tau) - \gamma \left[\dot{w}(\tau) + 2 \sum_{k=1}^{I(\tau)} (-1)^k \dot{w}(\tau - 2k) \right] - \varepsilon^4 w^3 \tag{2.10}$$
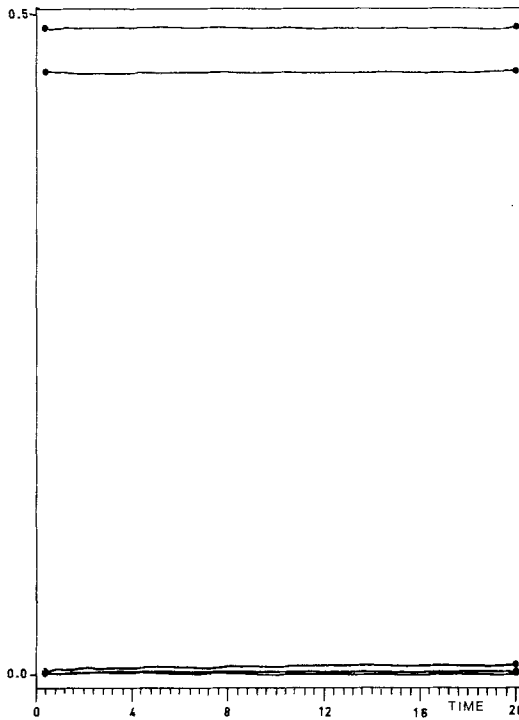
Fig. 1. Time behavior of $\langle \mathcal{E}_n \rangle_T$, $n = 3\text{--}17$ with $\varepsilon = 4$. Time is expressed in integration steps/10^6. Modes 9 and 7 are initially excited. At the right-hand side, from the top: modes 9, 7, 3, 5, and 11–17.

Having scaled our unknowns with the square root of the energy, the energy itself is replaced by a quantity scaled to the fixed value 1. More precisely, the new "energy" will satisfy the constraint

$$1 = \frac{1}{\gamma^4} \left[\frac{1}{2} \dot{w}^2 + \frac{\varepsilon^4}{4} w^4 + \sum_{n=1}^{\infty}{}' \frac{1}{2} (\dot{b}_n^2 + \tilde{\omega}_n^2 b_n^2) \right] \tag{2.11}$$

which is the rescaled form of Eq. (2.5). Consistently with Eq. (2.11), in the following by "energies" of the modes we mean the quantities

$$\mathscr{E}_n = (1/2\gamma^4)(\dot{b}_n^2 + \tilde{\omega}_n^2 b_n^2) = \tilde{E}_n/\tilde{E} = E_n/E \tag{2.12}$$

Of course, the initial conditions will take into account such constraint. For a fixed ε in Eqs. (2.10), the actual energy is obtained from Eq. (2.8b) once the parameters of the model (i.e., $\tilde{\alpha}$ and γ) have been fixed.
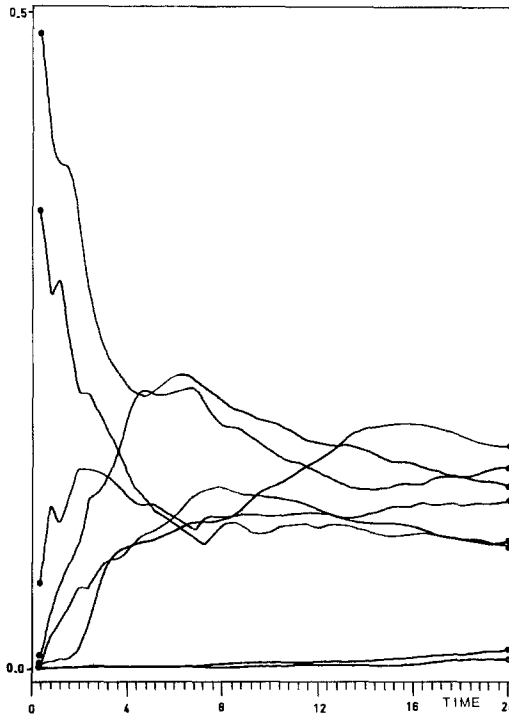

Fig. 2.   The same as in Fig. 1, with $\varepsilon = 10$. At the left-hand side and from the top: modes 9, 7, 3, 5, 13, 17, and 15. At the right-hand side: 13, 9, 5, 3, 11, 7, 15, and 17.

3. NUMERICAL EXPERIMENTS AND RESULTS

(i) Computer specifications: Most of the numerical experiments have been performed on CRAY X MP 12 and CRAY X MP 48 mainframes (CINECA, Bologna). Equations (2.10) have been integrated by a leapfrog method in double precision. A typical value for the integration step of the time τ is 0.0005. The scales of this "time" are fixed by the periods of the considered modes as given by the frequencies $\tilde{\omega}_n$ [see (2.7)]. In the worst case we have $2\pi/\tilde{\omega}_{25} = 0.16$ (about 320 steps per period). The maximal considered T consists of 2×10^7 steps, and corresponds to more than 2500 of the longest periods ($N = 1$). Actually, we have the same T as in ref. 5 but with an integration step ten times smaller [see also point (viii) below]. Several checks of stability have been performed.

(ii) Parameters: The value of τ has been chosen 0.2π (as in ref. 5, for the sake of comparison), but a number of experiments have been performed by varying this value, in order to show the role of the plate in the
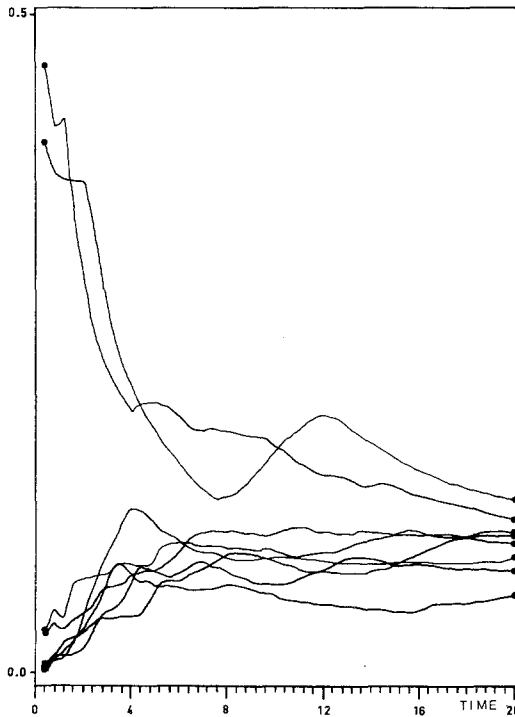

Fig. 3. The same as in Fig. 1, with $\varepsilon = 16$. At the left-hand side, from the top: modes 7, 9, 3, 5, and the others nearly together. At the right-hand side: 9, 7, 5, 17, 13, 15, 3, and 11.

sharing of energy. As to ε, it represents the order parameter with respect to which we investigated the model. We note that, according to definition (2.8b), variations of ε admit two interpretations: variations of initial energy $\tilde{E}$ at fixed nonlinearity $\tilde{\alpha}$, or variation of nonlinearity $\tilde{\alpha}$ at fixed energy $\tilde{E}$. Furthermore, ε^4 will represent directly $\tilde{E}$ for fixed $\tilde{\alpha}$ equal to γ^4 (resp. $\tilde{\alpha}$ for fixed $\tilde{E}$ equal to γ^4).

(iii)   Initial conditions: We have given 5% of the energy to the plate, and the remaining part equipartitioned among bunches of modes (up to nine, but typically two, with $N = 25$).

(iv)   Experiments: A first series of computations has been devoted to checking the trend to equipartition of energy in such a way as to recover the stochastic transition already studied by other authors and, with reference to it, to study the rate of energy exchange. Typical outputs are shown in Figs. 1–6, with the time averages (for $\langle \mathscr{E}_n \rangle_T$ or $\langle |\dot{\mathscr{E}}_n| \rangle_T$) versus time expressed in integration steps, for low and high energies, with a bunch
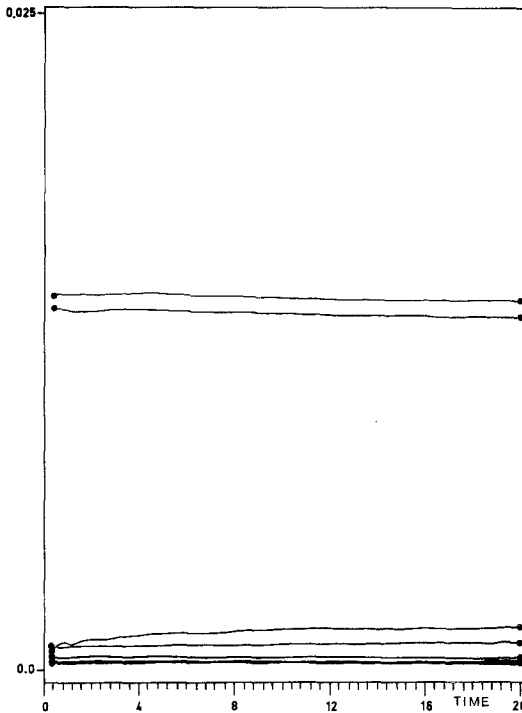

Fig. 4.   The same as in Fig. 1, for $\langle |\dot{\mathscr{E}}_n| \rangle_T$ with $\varepsilon = 4$. At the left-hand side, from the top: modes 9, 7, 5, 3, 11, and the others together. At the right-hand side: 13, 9, 3, 5, 11, 7, 15, and 17.

of two modes (7 and 9) initially excited. The high-energy cases clearly show a tendency which hereafter we shall recognize as "equipartition." At intermediate energy, such a tendency appears only for the lower modes: for these, the sharing of available energy is faster.

An instructive way of collecting the results is shown in Fig. 7, where the distributions of $\langle \mathcal{E}_n \rangle_T$ at $T_{\max}$ are plotted versus ε, i.e., versus energy. This figure confirms that, *within a fixed time of observation*, equipartition, up to a certain mode takes place only for a sufficiently large energy. In these conditions the energy works therefore as a critical parameter, roughly defining a threshold of stochasticity. Now, it results from the corresponding Figs. 4–6 and 8 that *the same happens for* $\langle |\dot{\mathcal{E}}_n| \rangle_T$. In the same sense as the extrapolated behavior for $T \to \infty$ lead to (1.2) for nonlinear chains, one obtains now that $\langle |\dot{\mathcal{E}}_n| \rangle = \text{const}$, which, for nonscaled variables, may be written as

$$\langle |\dot{E}_n| \rangle = \text{const}, \qquad \langle |\dot{J}_n| \rangle \propto 1/\omega_n \qquad (3.1)$$
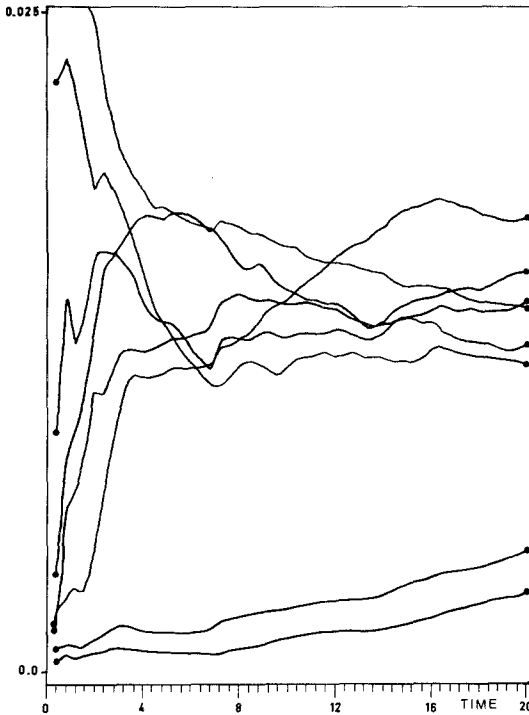

Fig. 5.   The same as in Fig. 4, with $\varepsilon = 10$. At the left-hand side, from the top: modes 9, 7, 3, 5, 13, 11, 15, and 17. At the right-hand side: 13, 9, 3, 5, 11, 7, 15, and 17.

This constitutes our main result, as mentioned before in (1.3). The rate of energy exchange is therefore sensitive to the stochastic transition, and could be used in turn as an indicator of stochasticity. Recalling that the same happens for one-dimensional chains, we stress two main differences: in that case (1) equipartition takes place for *actions*, and (2) the phenomenon regards the nonlinear chain as a whole, not a group of modes up to a certain N.

(v) In system (2.10) the equations for the modes b_n can be solved separately after solving the equation for w. Therefore there is no cutoff approximation in the analysis of the first N modes. We remark that higher and higher modes would appear in all the figures with flatter and flatter curves. For Figs. 2 and 3 and Figs. 5 and 6 in particular, the first omitted modes begin to be excited and contribute to the energy balance. Such a fact should be taken into account in speaking of equipartition at high values of ε in Figs. 7 and 8.
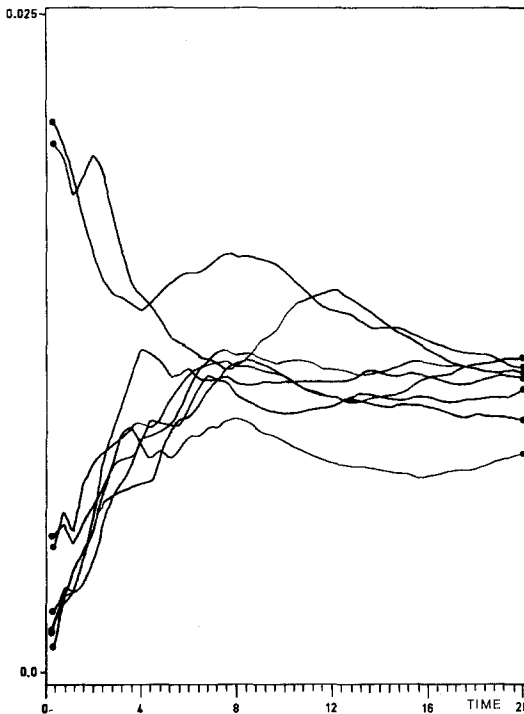

Fig. 6.   The same as in Fig. 4, with $\varepsilon = 16$. At the left-hand side, from the top: modes 7, 9, 5, 3, 13, 15, 11, and 17. At the right-hand side: 17, 7, 9 and 13, 5, 15, 3, 11.

(vi)   The influence of the other parameter γ has been studied and gives the expected results. Since γ represents the strength of the interaction between the plate and the field modes, an increase of γ increases the energy exchanges between the modes, with the same qualitative pattern as with an increase of energy (through ε) at fixed γ. This behavior has been checked with values of γ double and half of the reference value.

(vii)   Several experiments have been performed by varying the initial center of the bunch of excited modes, and the number of initially excited modes: these changes of the initial conditions confirm the results of point (iv), with obvious variations in the different "inertia" to the thermalization of the frequencies.

(viii)   We have done some "long" calculations with 10^8 steps, i.e., five times the previous ones. The trends of the curves are confirmed (see Figs. 9 and 10).
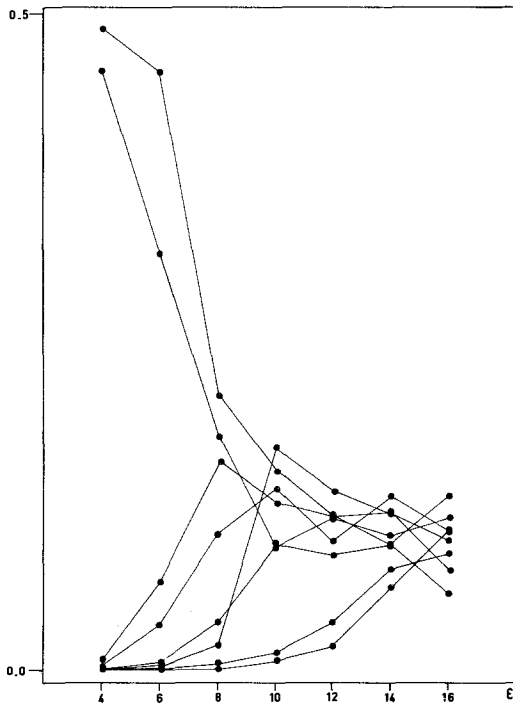

Fig. 7.   Values of $\langle \mathscr{E}_n \rangle_T$ at final T plotted versus ε. The role of ε in the thermalization process is clearly shown. The wider spreading at $\varepsilon = 16$ is only apparent, due to the increasing influence of the higher modes not drawn in the figure.

4. COMMENTS AND CONCLUSIONS

The results described above lead to the following conclusions.

1.   Exchanges of energy between modes are confirmed to be strictly related to the dynamical regime of motion, as in one-dimensional chains already studied. Indeed, within a fixed time, one observes a similar trend to equipartition for the energies and their exchanges.

2.   We have an answer to the initial problem about the possible relation between the equilibrium value of energy exchanges and the spectra: while in one-dimensional chains there was the extrapolated dependence $\langle|\dot{E}_n|\rangle \propto \omega_n$ in the limit $T \to \infty$, we have now $\langle|\dot{E}_n|\rangle = \text{const}$ in the same sense. The physical paradox implied by relation (1.2) for systems with unbounded spectra is solved by the fact that (1.2) is replaced in the present case by (3.1). It would be an interesting point to find out if this new "equipartition" relation is of general validity in the case of unbounded spectra, as we tend to believe, or if it is just a feature of the particular model studied here.
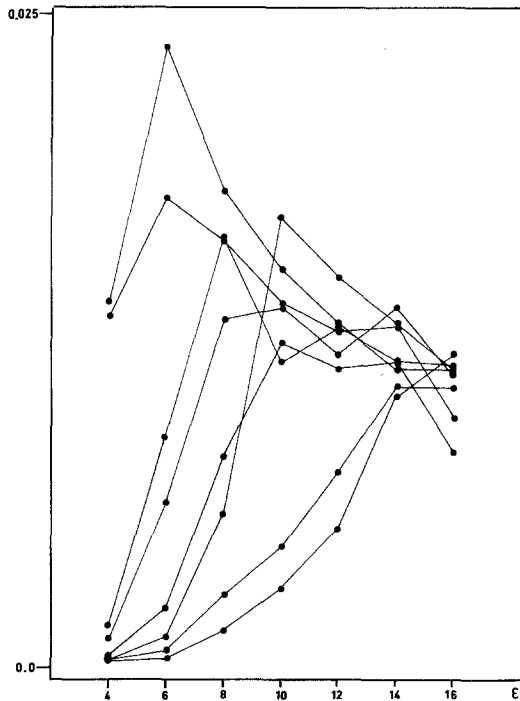

Fig. 8.   Values of $\langle|\dot{\mathscr{E}}_n|\rangle_T$ at final T plotted versus ε. The remarks in the legend to Fig. 7 apply also here.

3. The results may be rephrased in a geometrical language, recalling that the averaged velocities of the "actions," given by $\langle|\dot{E}_n|\rangle/\omega_n$, are proportional to $1/\omega_n$, and that therefore the equipartition of energy exchanges implies higher and higher "rigidity" for higher and higher modes. We speak of rigidity because, for unperturbed oscillators, the absolute value of the nth action variable gives the nth radius of the invariant torus, and the rate of change of this radius roughly represents its instability. We stress the fact that all this refers to the stochastic regime, and not to the ordered one.

As recalled in the Introduction, for nonlinear chains it is possible to distinguish between order, low stochasticity, and high stochasticity, just referring to the degree of complexity of orbits in the phase space and to the possibility (in the low stochastic regime) of the appearance of a trapping effect. The latter essentially consists, at a fixed energy, in a strong dependence of the relaxation time on the initial conditions. Turning to our
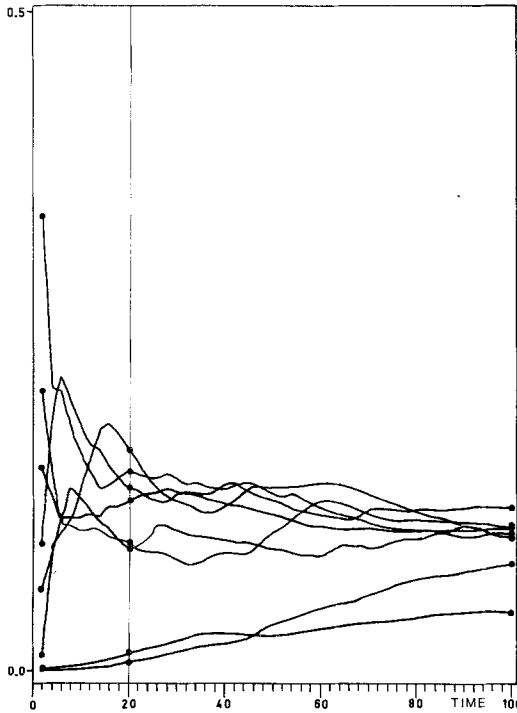

Fig. 9. Long calculations. Time behavior of $\langle\mathscr{E}_n\rangle_T$, $n = 3$–17 with $\varepsilon = 10$. Time is expressed in integration steps/2×10^6. The same as in Fig. 2 with the time of integration five times longer. The intermediate dots correspond to the final data of Fig. 2.

system, it is possible on one side to assert that a high stochasticization in the sense of nonlinear chains is impossible, because of (3.1), which "freezes" the high modes also in their thermalized phase; therefore the equipartition situation should be classified as a low stochastic one. But, on the other side, it is now particularly difficult to give evidence of trapping phenomena and of correlations with direct geometrical features of the orbits: both these subjects seem indeed to require a prohibitive amount of computations.

4. Points 1 and 2 depend strongly on the choice of various parameters [as said in point (vi) above], but the most delicate element is the total time of observation. The figures illustrate, in general, that a shorter T could lead to different conclusions, in the sense that the trend to equipartition could not be recognized as such. One may reasonably expect that for larger T more and more frozen modes will be excited to equipartition at a lower value (this expectation is confirmed by "long" calculations; see Figs. 9 and 10). An obvious question arises: for a fixed energy and a fixed number of modes, does there exist a time T such that equipartition (both in energy and rate of exchanges) is reached within T? If yes, this
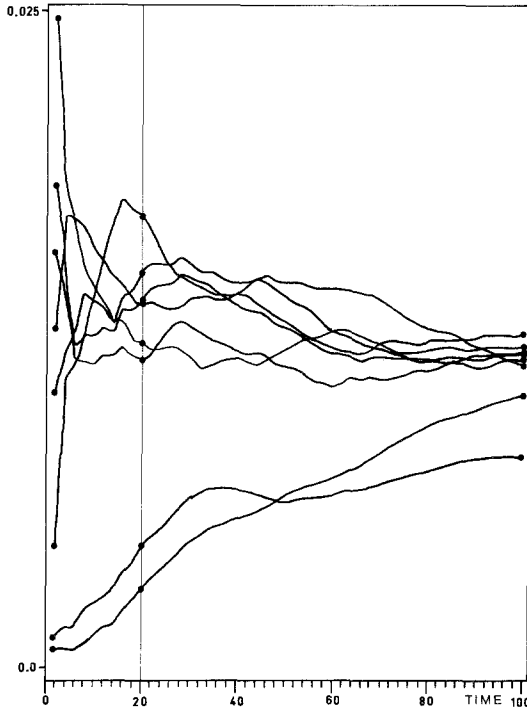

Fig. 10. Long calculations. The same as in Fig. 9, for $\langle |\dot{\mathscr{e}}_n| \rangle_T$. To be compared with Fig. 5.

would imply that every finite amount of energy tends to be shared, in a *sufficiently long* time, by the field as a whole. Another possibility is that, for the nth mode, there exist a critical value E_n^c such that for *every* time T the mode cannot be thermalized if the total available energy is less than E_n^c. Our computations cannot give a definite answer, but strongly support the evidence of a very fast growth of the relaxation time for thermalization. Once again, we remark that this problem is distinct from the one discussed in point 3, referring to already thermalized situations inasmuch the constant of (3.1) is different from zero.

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